A Vernacular for Linear Latent Growth Models

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In its most basic form, latent growth modeling (latent curve analysis) allows an assessment of individuals’ change in a measured variable $X$ over time. For simple linear models, as with other growth models, parameter estimates associated with the $\alpha$ construct (amount of $X$ at a chosen temporal reference point) and $\beta$ construct (growth in $X$ per unit time) are not invariant with respect to choice of reference point and time unit. Latent means, variances, and covariances change with different temporal metrics. This article offers a nomenclature for describing linear latent growth models, demonstrates how latent moment parameter estimates vary as a function of changes in the $\alpha$ reference point and in the $\beta$ growth metric, and presents a set of useful scale-free statistics for describing the results of linear latent growth modeling. Three examples are presented, two with a measured outcome and one with a latent outcome, and implications for applied and methodological researchers are presented.

Latent growth modeling (latent curve analysis), a structural equation modeling approach to understanding individuals’ longitudinal growth, has flourished in both application and methodological advances over the last decade or so, as evidenced by a quick perusal of any leading journal addressing human development or social science methodology. Models may address change in a measured outcome variable, change in a latent outcome variable, concurrent change in multiple domains, linear or nonlinear change, latent growth mixtures, accelerated longitudinal designs, and so forth (see Duncan, Duncan, Strycker, Li, & Alpert, 1999, for a thorough treatment of basic and advanced applications). Fundamental to all of the advanced growth modeling extensions, however, is the linear latent growth model examining change in a single measured variable over time. It is this simplest model that serves as the focus of this article, as enhancing our understanding of the information contained therein has the potential to inform all more complex extensions.

Specifically, the purpose of this article is to establish a means for communicating about linear latent growth models, both in terms of the structure of the models im-

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posed as well as regarding the results of the application of those models to sample data. Concerning the former, a simple symbolic notation will be proposed to communicate a model’s reference point for the $\alpha$ factor (e.g., initial time point) and the growth metric for the $\beta$ factor (e.g., annual change). This notation can be used by applied researchers to describe the structure of proposed theoretical models of interest, as well as by methodological researchers relating, say, model types manipulated as in a growth modeling simulation study. Then, via the proposed representational scheme, these designated $\alpha$ and $\beta$ factor characteristics are linked to the first and second latent moment parameter estimates arising in the application to sample data (i.e., estimated factor means, variances, and covariances). In turn, the notation and its relation to latent parameters will be used to derive a family of statistics that can concisely communicate the results of a linear modeling application. In particular, and perhaps most important, the nature of participants’ linear latent growth trajectories can be communicated using three statistics that are independent of the outcome variable’s units, of the $\alpha$ factor reference point, and of the $\beta$ factor growth metric. These proposed statistics will have immediate utility for applied researchers seeking to communicate a more thorough description of the developmental implications of their model’s results, and methodologists might find these statistics useful to represent data conditions to be manipulated within growth modeling simulation research. Thus, this work seeks to provide a comprehensive vernacular for linear latent growth models that will serve both applied and methodological researchers.

**DESCRIPTING IMPOSED LINEAR MODELS**

Briefly, in latent growth curve modeling a model investigating longitudinal growth in a variable $X$ may be expressed as $x = \Lambda \xi + \delta$. The vector $x$ contains values of $X$ across $p$ time points, $\Lambda$ is a matrix of loadings reflecting the hypothesized growth pattern underlying $X$, $\xi$ is a vector of factors (constructs) capturing the facets of growth being modeled, and $\delta$ is a vector of random normal residuals. For linear growth models, the focus of this article, the most common example posits $\xi = [\alpha \beta]'$ where $\alpha$ is an intercept factor representing individuals’ true amount of $X$ at the initial measured time point and $\beta$ is a slope factor representing their true rate of change per unit time. Given the expectation $E[\delta] = \mathbf{0}$, the model-implied first moment is $\mu = E[x] = \Lambda E[\xi] = \Lambda \kappa$, where $\kappa$ is a vector of factor means. As for the model-implied second moment $\Sigma$, assuming that $E[(\xi - \kappa)\delta'] = 0$ leads to $\Sigma = E[(x - \mu)(x - \mu)'] = \Lambda \Phi \Lambda' + \Theta$, where $\Phi$ is the covariance matrix of the factors in $\xi$ and $\Theta$ is the covariance matrix of the residuals in $\delta$. Readers desiring more information about latent growth models are referred to any of a number of readable pieces on the topic (e.g., Duncan & Duncan, 1995; Duncan, Duncan, & Stoolmiller, 1994; Duncan et al., 1999; Lawrence & Hancock, 1998; McArdle, 1988; Meredith, 1991; Meredith & Tisak, 1990; Moskowitz & Hershberger, 2002; Rogosa & Willett, 1985; Short, Horn, & McArdle, 1984; Stoolmiller, 1994, 1995; Tisak & Meredith, 1990; Willett & Sayer, 1994).
For linear growth models, the researcher may choose any reference point for the \( \alpha \) factor as well as any metric for the growth factor \( \beta \). The \( \alpha \) factor could represent the true amount of \( X \) at the initial point of measurement (as is most commonly done), at the final point of measurement, or at any point in between or outside that time frame as per the researcher’s preference. The \( \beta \) factor could be scaled to represent the rate of growth per week, per month, per year, or any other meaningful interval, regardless of the actual times of measurement. Setting the desired reference point for \( \alpha \) and the growth metric for \( \beta \) is accomplished by specifying the loading matrix \( \Lambda \), whose choice will not affect data–model fit but will affect growth parameter estimates (Mehta & West, 2000; Rogosa & Willett, 1985; Stoolmiller, 1995), parameter estimates of the impact of external predictors on growth factors (Stoel & van den Wittenboer, 2003), and parameter estimate standard errors (Biesanz, Deeb-Sossa, Papadakis, Bollen, & Curran, 2004). For example, consider \( p = 4 \) annual measurements (e.g., taken in 7th–10th grades), where the initial time point is the desired reference for \( \alpha \) and where an annual rate of change is the most useful metric for \( \beta \); the loading matrix is then the familiar

\[
\Lambda = \begin{bmatrix}
1 & 0 \\
1 & 1 \\
1 & 2 \\
1 & 3
\end{bmatrix}
\]

(1)

Had a time point 2 years after the final measurement (e.g., 12th grade) been the desired reference point for \( \alpha \), and had the desired growth metric for \( \beta \) been change per half-year (e.g., semester), then

\[
\Lambda = \begin{bmatrix}
1 & -10 \\
1 & -8 \\
1 & -6 \\
1 & -4
\end{bmatrix}
\]

(2)

For linear models with \( p \) measurements on the \( X \) variable, where those measurements are taken at equal intervals, any \( j \)th choice of the loading matrix \( \Lambda \) may be expressed as:

\[
\Lambda_j = \begin{bmatrix}
1 & \lambda_1 \\
1 & \lambda_2 \\
\vdots & \vdots \\
1 & \lambda_{p-1} \\
1 & \lambda_p
\end{bmatrix} = \begin{bmatrix}
1 & a_j + (0)b_j \\
1 & a_j + (1)b_j \\
\vdots & \vdots \\
1 & a_j + (p-2)b_j \\
1 & a_j + (p-1)b_j
\end{bmatrix} = \begin{bmatrix}1 & a_j \\ 0 & b_j\end{bmatrix} = \Lambda_0 C_j
\]

(3)
where loading matrix $\Lambda_0$ will be referred to as the standard baseline loading matrix, and where the matrix $C_j$ is the baseline transformation matrix that transforms $\Lambda_0$ to the desired loadings $\Lambda_j$. The matrix $C_j$ contains the baseline shift coefficient $a_j$ and the baseline scaling coefficient $b_j$. The coefficient $a_j$ is simply the first loading $\lambda_1$ in $\Lambda_j$; equivalently, it represents the degree of shift of the reference point for the $\alpha$ factor from the $j$th model back to the baseline model. With regard to this latter interpretation, $a_j = -10$ implies that the baseline model has its $\alpha$ factor reference point 10 growth units below that of the $j$th model. Thus, $a_j < 0$ means the baseline model’s $\alpha$ factor reference point is lower than that of the $j$th model, whereas $a_j > 0$ means the baseline model’s $\alpha$ factor reference point is higher than that of the $j$th model. Regarding the coefficient $b_j$, it can be interpreted most simply as the number of growth units of interest occurring between the first and second measurements, $\lambda_2 - \lambda_1$; equivalently, it represents the multiplicative degree of stretching or shrinking of the $j$th model’s $\beta$ factor growth metric required to make it consistent with that of the baseline model. With regard to this latter interpretation, $b_j = 2$ implies that the baseline model has a growth metric for the $\beta$ factor that is twice the duration of that in the $j$th model. Thus, $b_j < 1$ means the baseline model’s $\beta$ factor has a temporally shorter growth metric than the $j$th model, whereas $b_j > 1$ means the baseline model’s $\beta$ factor has a temporally longer growth metric than the $j$th model.

The most common linear growth model, mentioned previously, is where $\Lambda_j = \Lambda_0$, thus having $a_j = 0$ and $b_j = 1$ (i.e., $C_j = I$). This standard baseline linear growth model may simply be referred to as a $\{0, 1\}$ model, corresponding to $\{a_j, b_j\}$. As for the example associated with Equation 2, setting the growth rate metric to semi-annual across the $p = 4$ annual measurements makes the first (and each adjacent) interval equal to two units of theoretical interest; hence, $b_j = 2$. The reference point for the $\alpha$ factor was also shifted relative to a baseline $\{0, 1\}$ model up to a point that is 5 years forward. Put differently, the initial time point actually observed in the data (also, the baseline model’s reference point) is 10 half-years below the $j$th model’s $\alpha$ factor reference point; thus, $a_j = -10$. This information is communicated quite concisely by referring to that imposed linear latent growth model as a $\{-10, 2\}$ model.

Of course, the $p$ measurements analyzed with linear latent growth models need not be made at equal intervals. A more general expression for $\Lambda_j$ is as follows:

$$
\Lambda = \begin{bmatrix}
1 & \lambda_1 \\
1 & \lambda_2 \\
1 & \lambda_3 \\
\vdots & \vdots \\
1 & \lambda_{p-1} \\
1 & \lambda_p
\end{bmatrix} = \begin{bmatrix}
1 & a_j + (0)b_j \\
1 & a_j + (1)b_j \\
1 & a_j + (\omega_2)b_j \\
\vdots & \vdots \\
1 & a_j + (\omega_{p-1})b_j \\
1 & a_j + (\omega_p)b_j
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
1 & 1 \\
1 & \omega_3 \\
\vdots & \vdots \\
1 & \omega_{p-1} \\
1 & \omega_p
\end{bmatrix} \begin{bmatrix}
a_j \\
b_j
\end{bmatrix} = \Omega_0 C_j
$$

(4)
where the loadings of interest in $\Lambda_j (\lambda_1 < \lambda_2 < \ldots < \lambda_p)$ are simply a linear function of a baseline set of loadings in the general baseline loading matrix $\Omega_0 (0 < \omega_1 < \omega_2 < \ldots < \omega_p)$. Note that $p$ equally spaced measurements in the second row of the standard baseline loading matrix $\Lambda_0, [0 \ 1 \ \ldots \ \ p - 1]^\prime$, constitute a special case of $\Omega_0$. The baseline model in $\Omega_0$ may still be correctly designated as a $\{0, 1\}$ model, and the interpretation of $a_j$ and $b_j$, as well as the ability to use them to describe any linear latent growth model, remain unchanged.

Consider an unequally spaced example with $p = 6$ measurements, the first four spanning three semimonthly intervals and the last two following monthly intervals. The baseline $\{0, 1\}$ model would have values in the second column of $\Omega_0$ as $[0 \ 1 \ 2 \ 3 \ 5 \ 7]^\prime$, thus defining the growth metric as semimonthly. A model also with the initial measurement as a reference point for $\alpha$, but with the growth metric as monthly rather than semimonthly, would be described as $\{0, .5\}$ and thus would have the second column of $\Lambda_j$ as $[0 \ .5 \ 1.0 \ 1.5 \ 2.5 \ 3.5]^\prime$. Alternatively, a model with the final measurement as a reference point for $\alpha$, and invoking a monthly growth metric, would be described as $\{-3.5, .5\}$ and thus would have the second column of $\Lambda_j$ as $[-3.5 \ -3.0 \ -2.5 \ -2.0 \ -1.0 \ 0]^\prime$. A model with the fourth measurement as a reference point for $\alpha$, and utilizing a monthly growth metric, would be described as $\{-1.5, .5\}$ and thus would have the second column of $\Lambda_j$ as $[-1.5 \ -1.0 \ -0.5 \ 0 \ 1.0 \ 2.0]^\prime$. Finally, a model with the fourth measurement as a reference point for $\alpha$ and using a semimonthly growth metric would be described as $\{-3, 1\}$ and thus would have the second column of $\Lambda_j$ as $[-3 \ -2 \ -1 \ 0 \ 2 \ 4]^\prime$.

It should be noted that, in the very common scenario of $p$ equally spaced measurements, the $\{a_j, b_j\}$ designation provides a complete specification of the pattern of measurement points for the linear latent growth model. Without equally spaced measurements, however, such is not the case. For example, consider the final model just given with a designation of $\{-3, 1\}$, which corresponded to the second column of $\Lambda_j$ as $[-3 \ -2 \ -1 \ 0 \ 2 \ 4]^\prime$. The same $\{-3, 1\}$ designation could represent an equally spaced model with the second column of $\Lambda_j$ as $[-3 \ -2 \ -1 \ 0 \ 1 \ 2]^\prime$. It could also represent a model with varied spacing like $[-3.0 \ -2.0 \ -1.5 \ .5 \ 2.0 \ 3.0]^\prime$, such as measurements occurring at the beginning of Month 1, beginning of Month 2, middle of Month 3, middle of Month 4, beginning of Month 6, and beginning of Month 7. The purpose of the $\{a_j, b_j\}$ nomenclature, however, is not to convey the precise pattern of the measurement points. Rather, it is to describe the imposed linear latent growth model given the context of the specific $p$ measurements. Thus, for a researcher communicating that measurements were made at the previous $p = 6$ varied semimonthly and monthly time points, describing an imposed linear model as $\{-4, 2\}$ communicates the following. The $a_j = -4$ conveys that the initial time point measured is actually four growth units below the chosen reference point for $\alpha$. The $b_j = 2$ indicates that the interval between the first and second measurement is two times the de-
sired growth metric; if the second point is 1 month after the first, then the desired growth metric is semimonthly.

THE EFFECT OF \( \{a_j, b_j\} \) MODEL SPECIFICATION ON LATENT GROWTH PARAMETER ESTIMATES

Having proposed the \( \{a_j, b_j\} \) notational system for describing the structure of specific linear latent growth models, it becomes useful to understand the impact of a researcher’s choice of \( a_j \) and \( b_j \) on the first and second latent moments in the data. This section explicates these relations first in an attempt to provide a useful didactic consolidation drawing from prior related efforts (e.g., Biesanz et al., 2004; Mehta & West, 2000; Stoel & van den Wittenboer, 2003). Second, and more practically, the explication serves to facilitate the subsequent derivation of statistics useful for describing the application of desired models to specific sample data.

To start, when the baseline \( \{0, 1\} \) model is fit to a data set with measurements at \( p \) time points, it yields a vector of estimates of the first latent moments that may be denoted as \( \hat{\kappa}_0 \), with elements \( \hat{\kappa}_{00} \) and \( \hat{\kappa}_{01} \). Estimates of the second latent moments associated with the baseline \( \{0, 1\} \) model reside in what may be designated as matrix \( \hat{\Sigma}_0 \), with unique elements \( \hat{\sigma}^2_{00}, \hat{\sigma}^2_{01}, \) and \( \hat{\sigma}_{00\!1} \). Similarly, any \( j \)th model \( \{a_j, b_j\} \) or \( k \)th model \( \{a_k, b_k\} \) has estimated first latent moment vectors \( \hat{\kappa}_j \) and \( \hat{\kappa}_k \), and estimated second latent moment matrices \( \hat{\Sigma}_j \) and \( \hat{\Sigma}_k \), respectively.

As is well known (e.g., MacCallum, Wegener, Uchino, & Fabrigar, 1993; Meredith & Tisak, 1990), the data–model fit of any \( j \)th and \( k \)th linear models will be identical, yielding equivalent model-implied first moments (\( \hat{\mu}_j = \hat{\mu}_k \)) and second moments (\( \hat{\Sigma}_j = \hat{\Sigma}_k \)). With regard to the first latent moment, the model-implied mean vector for each model may be expressed as \( \hat{\mu}_j = \Lambda_j \hat{\kappa}_j \) and \( \hat{\mu}_k = \Lambda_k \hat{\kappa}_k \). The equivalence of \( \hat{\mu}_j = \hat{\mu}_k \) implies that \( \Lambda_j \hat{\kappa}_j = \Lambda_k \hat{\kappa}_k \). Referring each back to a baseline \( \{0, 1\} \) model with \( \Omega_0 \) in Equation 4, substitution yields \( (\Omega_0 C_j) \hat{\kappa}_j = (\Omega_0 C_k) \hat{\kappa}_k \). With invertible \( C_k \), rearranging and simplifying implies that \( \hat{\kappa}_k = (C_k^{-1} C_j) \hat{\kappa}_j \); that is, the relation between estimated factor means under two linear growth models is simply a function of their respective baseline transformation matrices \( C_j \) and \( C_k \). Thus, for a \( j \)th model \( \{a_j, b_j\} \) yielding latent mean estimates in \( \hat{\kappa}_j \), one can determine the parameter estimates in \( \hat{\kappa}_k \) for a \( k \)th model \( \{a_k, b_k\} \) without fitting the \( k \)th model. Given the definition for the baseline transformation matrix \( C_j \) from Equation 3, it follows that

\[
C_k^{-1} = \begin{bmatrix} 1 & \frac{-a_k}{b_k} \\ 0 & \frac{1}{b_k} \end{bmatrix} \quad \text{and} \quad C_j^{-1}C_k = \begin{bmatrix} 1 & \frac{a_j - (b_j / b_k) a_k}{b_j} \\ 0 & \frac{a_j - (b_j / b_k) a_k}{b_k} \end{bmatrix}
\]
Thus, the previous relation \( \hat{k}_k = (C_k^{-1}C_j)\hat{k}_j \) leads to

\[
\begin{bmatrix}
\hat{k}_{\alpha k} \\
\hat{k}_{\beta k}
\end{bmatrix} = 
\begin{bmatrix}
1 & a_j - \left( \frac{b_j}{b_k} \right) a_k \\
0 & \left( \frac{b_j}{b_k} \right)
\end{bmatrix} 
\begin{bmatrix}
\hat{k}_{\alpha j} \\
\hat{k}_{\beta j}
\end{bmatrix}
\]  

(6)

or more directly

\[
\hat{k}_{\alpha k} = \hat{k}_{\alpha j} + \left( a_j - \left( \frac{b_j}{b_k} \right) a_k \right) \hat{k}_{\beta j}
\]  

(7)

\[
\hat{k}_{\beta k} = \left( \frac{b_j}{b_k} \right) \hat{k}_{\beta j}
\]  

(8)

Additionally, had the \( j \)th model been the baseline, where \( \{ a_j, b_j \} = \{ 0, 1 \} \), the relations would simplify to

\[
\hat{k}_{\alpha k} = \hat{k}_{\alpha 0} - \left( \frac{a_k}{b_k} \right) \hat{k}_{\beta 0}
\]  

(9)

\[
\hat{k}_{\beta k} = \left( \frac{1}{b_k} \right) \hat{k}_{\beta 0}
\]  

(10)

Starting with the estimated slope factor mean, the practical implication for \( \hat{k}_\beta \) in Equation 8 is that it is affected only by rescaling from \( b_j \) to \( b_k \) (but not by shifting from \( a_j \) to \( a_k \)). As is intuitive, making the unit of time shorter (i.e., such that \( b_k > b_j \)) will decrease the magnitude of \( \hat{k}_\beta \) proportionally. For example, compared to an average monthly growth rate, the average semimonthly growth rate is half as large. Regarding the estimated intercept factor mean in Equation 7, \( \hat{k}_{\alpha} \) changes with both shifts in the reference point for \( \alpha \) as well as changes in the \( \beta \) factor growth metric. If the growth metric is unchanged (i.e., \( b_j = b_k \)), then \( \hat{k}_{\alpha} \) changes one unit of \( \hat{k}_{\beta j} \) for every unit shift in reference point. If the growth metric is changed as well, the value of the shift must first be rescaled before determining the number of units of \( \hat{k}_{\beta j} \) by which to adjust \( \hat{k}_{\alpha j} \). As a brief example, imagine a researcher changes from a baseline \( \{ 0, 1 \} \) model on \( p = 4 \) annual measurements to a \( \{-6, 2\} \) model in which the last (fourth) time point is the reference point for \( \alpha \) and the desired growth metric is semiannual. As the reader may verify using Equations 9 and 10, if originally \( \hat{k}_{\alpha j} = \hat{k}_{\alpha 0} = 100 \) and \( \hat{k}_{\beta j} = \hat{k}_{\beta 0} = 10 \), then changing to the \( \{-6, 2\} \) model configuration would yield \( \hat{k}_{\alpha k} = 130 \) and \( \hat{k}_{\beta k} = 5 \).
With regard to the second latent moments, the model-implied covariance matrices for the \( j \)th and \( k \)th models successfully and equivalently applied to the same data may be expressed as \( \Sigma_j = \Lambda_j \Phi_j \Lambda_j' + \Theta_j \) and \( \Sigma_k = \Lambda_k \Phi_k \Lambda_k' + \Theta_k \). Given that the two models are structurally isomorphic (and without additional error covariances), \( \Lambda_j \Phi_j \Lambda_j' = \Lambda_k \Phi_k \Lambda_k' \) (and, in fact, \( \Theta_j = \Theta_k \)). Referring each back to the baseline \{0, 1\} model with \( \Omega_0 \) in Equation 4, substitution yields \( (\Omega_0 C_j) \Phi_j (\Omega_0 C_j)' = (\Omega_0 C_k) \Phi_k (\Omega_0 C_k)' \). Rearranging and simplifying establishes \( \Phi_k = (C_k^{-1} C_j) \Phi_j (C_k^{-1} C_j)' \); that is, the relation between estimated factor (co)variance estimates under two linear latent growth models is simply a function of their respective baseline transformation matrices \( C_j \) and \( C_k \). Thus, for a \( j \)th model \{\( a_j, b_j \)\} yielding key latent parameter estimates in \( \Phi_j \), one can determine the parameter estimates in \( \Phi_k \) for a \( k \)th model \{\( a_k, b_k \)\} without fitting the \( k \)th model. Given the previous expression for \( C_k^{-1} C_j \), by substitution it follows that

\[
\hat{\Phi}_k = \begin{bmatrix}
\hat{\sigma}^2_{\alpha k} & \hat{\sigma}_{\alpha \beta k} \\
\hat{\sigma}_{\alpha \beta k} & \hat{\sigma}^2_{\beta k}
\end{bmatrix} = \begin{bmatrix}
a_j - \left( \frac{b_j}{b_k} \right) a_k & \hat{\sigma}^2_{\alpha j} + \hat{\sigma}_{\alpha \beta j} \\
\hat{\sigma}_{\alpha \beta j} & a_j - \left( \frac{b_j}{b_k} \right) a_k + \left( \frac{b_j}{b_k} \right) a_k
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]  

or more directly,

\[
\hat{\sigma}^2_{\alpha k} = \hat{\sigma}^2_{\alpha j} + 2 \left[ a_j - \left( \frac{b_j}{b_k} \right) a_k \right] \hat{\sigma}_{\alpha \beta j} + \left[ a_j - \left( \frac{b_j}{b_k} \right) a_k \right]^2 \hat{\sigma}^2_{\beta j}
\]  

\[
\hat{\sigma}_{\alpha \beta k} = \left( \frac{b_j}{b_k} \right) \hat{\sigma}_{\alpha \beta j} + \left( \frac{b_j}{b_k} \right) a_j - \left( \frac{b_j}{b_k} \right) a_k \hat{\sigma}^2_{\beta j}
\]

\[
\hat{\sigma}^2_{\beta k} = \left( \frac{b_j}{b_k} \right)^2 \hat{\sigma}^2_{\beta j}
\]

Additionally, had the \( j \)th model been the baseline, where \( \{a_j, b_j\} = \{0, 1\} \), the relations would simplify to

\[
\hat{\sigma}^2_{\alpha k} = \hat{\sigma}^2_{\alpha 0} - 2 \left( \frac{a_k}{b_k} \right) \hat{\sigma}_{\alpha \beta 0} + \left( \frac{a_k}{b_k} \right)^2 \hat{\sigma}^2_{\beta 0}
\]
First, with regard to shifting the intercept factor reference point by changing from \(a_j \) to \(a_k \), if the \(\beta\) factor growth metric is held constant (i.e., \(b_j = b_k\)) then \(\hat{\sigma}_\alpha^2\) and \(\hat{\sigma}_{\alpha\beta}\) will change whereas \(\hat{\sigma}_\beta^2\) will be unaffected. This makes sense given that subjects’ positions along their growth trajectory at this new reference point will likely vary differently and covary differently with their slopes, whereas the slopes of the linear trajectories (and hence the slope variance estimate \(\hat{\sigma}_\beta^2\)) are unchanged by the new reference point. Second, with regard to changes in \(b\), rescaling the growth metric alters \(\hat{\sigma}_\beta^2\) and \(\hat{\sigma}_{\alpha\beta}\) (as would be expected because both depend on the metric of the \(\beta\) factor). As for \(\hat{\sigma}_\alpha^2\), the effect of changes in \(b\) requires elaboration. If \(a_j = a_k = 0\), then changes in growth metric do not affect \(\hat{\sigma}_\alpha^2\). Otherwise, if \(a_j \neq a_k \neq 0\), then changes in the growth metric will have an impact. However, in practice, changes in growth metric are usually accompanied by changes in \(a\). For example, imagine a \([-1, 1]\) model for \(p = 4\) equally spaced time points, thus having the second column of \(\Lambda_j\) of \([-1 0 1 2]'\). Halving the growth metric (e.g., changing from weekly to semiweekly) leads to a second column of \(\Lambda_j\) as \([-2 0 2 4]'\), which is a \([-2, 2]\) model—not a \([-1, 2]\) model. That is, the value of \(a\) has changed proportionally as well, which would be expected more often in practice rather than holding \(a\) constant (except when \(a_j = a_k = 0\)). In general, if \(\{a_j, b_k\} = \{ca_j, cb_k\}\) where \(c \neq 0\), then \(\hat{\sigma}_\alpha^2\) is unaltered by changes in \(b\). Finally, recall the previous example where a researcher changed from a baseline \([0, 1]\) model on \(p = 4\) annual measurements to a \([-6, 2]\) model. As the reader may verify using Equations 15 through 17, if originally \(\hat{\sigma}_{a_j}^2 = \hat{\sigma}_{a_0}^2 = 40\), \(\hat{\sigma}_{b_i}^2 = \hat{\sigma}_{b_{p0}}^2 = 10\), and \(\hat{\sigma}_{\alpha\beta, i} = \hat{\sigma}_{\alpha\beta, 0} = 10\), then the final estimates after shifting \(a\) and rescaling \(b\) will be \(\hat{\sigma}_{a_k}^2 = 190\), \(\hat{\sigma}_{\beta_k}^2 = 2.5\), and \(\hat{\sigma}_{\alpha\beta_k} = 20\). As mentioned previously, the choice of \(\{a_j, b_k\}\) model specification on latent growth parameter estimates can be extended to the effect of external predictors on growth factors, to standard errors of parameter estimates, and even the case of polynomial growth; see Biesanz et al. (2004) for more detailed discussion.

**DESCRIBING RESULTS OF LINEAR LATENT GROWTH MODELING**

Having described a simple way of designating linear latent growth models to be imposed on data, and having illustrated the impact of model choice on the latent
structure parameters, next a series of useful statistics for describing the results of a well-fitting linear latent growth model is presented.

The Aperture

When $\hat{\sigma}_{\beta}^2 > 0$, a collection of unbounded growth trajectory lines resembles a bowtie, either horizontal or askew, where the lines appear to fan out from the bowtie’s knot (also termed *fan-spread*; see, e.g., Rogosa & Willett, 1985). This knot represents the location of the lines’ relative (or even complete) convergence, at which the conditional variance of those lines is minimized. Similarly, one can imagine the growth trajectory lines resembling light rays passing through the opening in a camera and becoming inverted once inside. Drawing from this second analogy, the point of the trajectories’ relative convergence will be referred to as the *aperture*. The aperture, like the bowtie’s knot, represents the point in time where individuals are estimated to be the most similar in terms of their true amount of $X$. This point may occur anywhere inside or outside of the time span of the $p$ measurements. If the aperture occurs at or below the initial measurement, the lines of growth trajectory within the measured time span resemble some portion of the right side of the bowtie (diverging over the interval away from the aperture). If the aperture occurs at or above the $p$th measurement, the lines of growth trajectory within the measured time span resemble some portion of the left side of the bowtie (converging over the interval toward the aperture).

The aperture may be most usefully operationalized in terms of the $\alpha$ factor. Specifically, when $\hat{\sigma}_{\beta}^2 > 0$, the sample’s aperture may be defined as the $\alpha$ factor reference point for the model $\{a_j^*, b_j\}$ that minimizes the intercept factor variance to $\hat{\sigma}_{a_j^*}^2 \geq 0$. The coefficient $a_j^*$ is the *aperture shift coefficient* for the sample, which may be determined as follows. Drawing from Equation 12, the following function relates $\hat{\sigma}_{\alpha_k}^2$ to shifts in the $\alpha$ factor reference point from $a_j$ to some $a_k$ (holding growth metric constant, $b_j = b_k$):

$$\hat{\sigma}_{\alpha_k}^2 = \hat{\sigma}_{\alpha_j}^2 + 2(a_j - a_k)\hat{\sigma}_{\alpha\beta} + (a_j - a_k)^2 \hat{\sigma}_{\beta}^2$$  \hspace{1cm} (18)

Differentiating Equation 18 with respect to $a_k$, the reader may verify that the sample’s aperture shift coefficient $a_j^*$ may be determined as

$$a_j^* = a_j + \frac{\hat{\sigma}_{\alpha\beta}}{\hat{\sigma}_{\beta}^2}$$  \hspace{1cm} (19)

when $\hat{\sigma}_{\beta}^2 > 0$.

Regarding the location of the aperture relative to an $\{a_j, b_j\}$ model, if $a_j^* = a_j$ then the aperture is already at the reference point chosen for the $j$th model’s $\alpha$ fac-
tor. If $a_j^* < a_j$, then the aperture is above the $j$th model’s $\alpha$ reference point. If $a_j^* > a_j$, then the aperture is below the $j$th model’s $\alpha$ reference point. For example, relative to a $\{-6, 2\}$ model, $a_j^* = -9$ indicates that the aperture occurs three metric growth units above the $j$th model’s $\alpha$ reference point. More usefully, an index may be derived to frame the location of the aperture relative to the span of the $p$ measured points. Within the context of an $\{a_j, b_j\}$ model with loading matrix $\Lambda_j$, the sample’s relative aperture location (RAL) can be determined as

$$\frac{-a_j^*}{\lambda_p - \lambda_1}$$

Values below 0 indicate that the aperture is below the measured time span, whereas a value equal to 0 implies that the aperture falls precisely at the initial measurement point. Values between 0 and 1 indicate that the aperture falls within the measured time span, and proportionally where within that span (e.g., 0.20 implies that the aperture occurs after 20% of the total time interval has passed). A value equal to 1 implies that the aperture falls precisely at the final ($p$th) measurement point, whereas values above 1 indicate that the aperture is above the measured time span.

Note that using the baseline $\{0, 1\}$ model with loading matrix $\Omega_0$ as a frame of reference, and assuming $\hat{\sigma}^2_{\beta_0} > 0$, the sample’s baseline aperture may be defined as the reference point for the $\alpha$ factor in the model $\{a_0^*, 1\}$ such that the baseline aperture shift coefficient $a_0^*$ minimizes the $\alpha$ factor variance to $\hat{\sigma}^2_{a_0^*} \geq 0$. Following from Equation 19,

$$a_0^* = \frac{\hat{\sigma}_{\alpha 0}}{\hat{\sigma}^2_{\beta_0}}$$

when $\hat{\sigma}^2_{\beta_0} > 0$. For example, a value of $a_0^* = 2.3$ indicates that the baseline aperture occurs 2.3 growth units below the initial measurement. Further, a relative aperture location may be defined for the baseline model as

$$\frac{-a_0^*}{\lambda_p - \lambda_1}$$

which would be identical to that determined using any $\{a_j, b_j\}$ model as a frame of reference (although the corresponding loadings in the denominator of Equations 20 and 22 could differ).

Practically speaking, the usefulness of locating the aperture is anticipated to be twofold. First, as mentioned previously, the aperture represents the point in time when individuals are estimated to be most similar in their true amount of $X$. For developmental researchers, approximating this juncture could hold great value. The practitioner interested in interventions, for example, might find that estimating at
what time point individuals are most homogeneous is useful for targeting an inter-
vvention, thereby maximizing its effectiveness. Had individuals been more diverse, as expected at surrounding time points, the intervention might be less effective given the range of individual developmental levels present at those points. Caution is certainly warranted, however, when the aperture is outside the measurement inter-
val. Although a linear model may satisfactorily characterize the developmental process within the time interval studied, such may not be the case outside of that interval. Even if a linear model were reasonable beyond the interval, the aperture’s location could be meaningless or without practical value. In the case where an evaluation of a program or a treatment intervention is already underway, an aperture estimated to have occurred prior to the period of evaluation or intervention may hold little utility. Hence the aperture, and the statistics related to this notion, rest on the meaningfulness of the aperture within the researcher’s field of study.

A second anticipated use for the aperture, as developed later, is that it can serve as the foundation for characterizing any particular linear growth process. Simply knowing the RAL provides an initial description of the growth trajectories (e.g., resembling the right side of a bowtie). In concert with additional measures described next, a complete description of individuals’ growth trajectories in a given sample is facilitated.

Latent Moments at the Aperture

Latent parameter estimates for models with the aperture as the $\alpha$ factor reference point are as follows, starting with the first moments. Relative to an $\{a_j, b_j\}$ model, it follows from Equation 7 that an $\{a_j^*, b_j\}$ model has a latent mean estimate for the intercept factor $\alpha$ of

$$\hat{k}_{\alpha,j} = \hat{k}_{\alpha,j} + (a_j - a_j^*)\hat{k}_{\beta,j}$$

estimating the mean amount of the variable $X$ at the sample’s aperture. Meanwhile, Equation 8 implies that the $\{a_j^*, b_j\}$ model has same latent mean estimate $\hat{k}_{\beta,j}$ for the slope factor $\beta$ in the chosen growth metric. Further, had the $j$th model been the baseline $\{0, 1\}$ model, it follows directly that an $\{a_0^*, 1\}$ model would have a latent mean estimate for the intercept factor $\alpha$ of

$$\hat{k}_{\alpha,0} = \hat{k}_{\alpha,0} - (a_0^*)\hat{k}_{\beta,0}$$

Note that $\hat{k}_{\alpha,0} = \hat{k}_{\alpha,j}^*$, with each representing the estimated mean amount of the variable $X$ at the sample’s aperture. Meanwhile, as per Equation 10, the $\{a_0^*, 1\}$ model has the same latent mean estimate $\hat{k}_{\beta,0}$ for the slope factor $\beta$ as in the baseline growth model.
Consider now the estimates of the second latent moments. Relative to an \( \{a_j, b_j\} \) model, an \( \{a_j^*, b_j\} \) model has latent (co)variance estimates as follows. As the reader may verify, substitution of the Equation 19 expression for \( a_j^* \) into Equations 12 through 14 yields

\[
\hat{\sigma}^2_{a_j^*} = \hat{\sigma}^2_{a_j} - \frac{(\hat{\sigma}_{a\beta_j})^2}{\hat{\sigma}^2_{b_j}} \tag{25}
\]

\[
\hat{\sigma}_{a\beta_j^*} = 0 \tag{26}
\]

\[
\hat{\sigma}^2_{b_j^*} = \hat{\sigma}^2_{b_j} \tag{27}
\]

Had the \( j \)th model been the baseline \( \{0, 1\} \) model, it follows directly that the second latent moment estimates for the \( \{a_0^*, 1\} \) model would be

\[
\hat{\sigma}^2_{a_0^*} = \hat{\sigma}^2_{a_0} - \frac{(\hat{\sigma}_{a\beta_0})^2}{\hat{\sigma}^2_{b_0}} \tag{28}
\]

\[
\hat{\sigma}_{a\beta_0^*} = 0 \tag{29}
\]

\[
\hat{\sigma}^2_{b_0^*} = \hat{\sigma}^2_{b_0} \tag{30}
\]

First, from a practical standpoint, the average and the variability of individuals’ positions along their growth trajectories at the point of aperture may be conveyed respectively by two statistics, \( \hat{k}_{a_j^*} \) and \( \hat{\sigma}^2_{a_j^*} \) (or \( \hat{\sigma}_{a_j^*} \)). Whereas the former statistic may be regarded as a theoretical average either of \( X \) values or of true values underlying \( X \) (that follow the linear trajectory), the latter statistic only applies to true values on the trajectories. That is, the variance \( \hat{\sigma}^2_{a_j^*} \) does not contain expected error variability in observed \( X \) values that might occur at the point of aperture. Rather, \( \hat{\sigma}^2_{a_j^*} \) it is an estimate reflecting individuals’ true variability when they are at their most homogeneous developmental juncture.

Second, it is apparent from the derived relations that the sample’s aperture defines not just a point of minimum variance in individuals’ positions along their true growth trajectories (Mehta & West, 2000; Rogosa & Willett, 1985), but also a point where intercept and slope do not covary (see also Equations 26 and 29; Mehta & West, 2000). Following similar derivations, the reader can show that as reference points for the \( \alpha \) factor shift to points farther below the sample’s aper-
ture the covariance between the intercept and slope factors becomes increasingly negative; conversely, as reference points for the $\alpha$ factor shift to points farther above the sample’s aperture the covariance between the intercept and slope factors becomes increasingly positive. Put more visually, for time points on the left side of the bowtie’s knot where the covariance is negative, trajectory lines at higher levels generally have a larger negative (or smaller positive) slope and lines at lower levels generally have a smaller negative (or larger positive) slope. On the other hand, for time points on the right side of the bowtie’s knot where the covariance is positive, trajectory lines at higher levels generally have a larger positive (or smaller negative) slope and lines at lower levels generally have a larger negative (or smaller positive) slope. This result also implies that researchers’ findings regarding the covariance between latent intercept and slope factors is directly related to the location of their chosen reference point relative to the sample’s aperture. Negative covariance implies the $\alpha$ factor reference point is to the left relative to the aperture (i.e., earlier in time), whereas positive covariance means the reference point is to the right relative to the aperture (i.e., later in time).

Additional Descriptive Indexes

The previous section proposed several potentially useful statistics, including the RAL and mean and variance estimates at the aperture. The RAL provides a scale-free description of what part of the bowtie is represented in the data; meanwhile, the suggested descriptives of $\hat{\kappa}_{\alpha, j}^*$ and $\hat{\sigma}_{\alpha, j}^2$ (or $\hat{\sigma}_{\alpha, j}^*$) relate directly to the nature of the outcome variable at the aperture. In this section two additional scale-free descriptives are offered to facilitate a more complete description of the trajectories in the data at hand, the first dealing further with the degree of convergence reached at the aperture (i.e., the relative vertical thickness of the bowtie’s knot) and the second addressing the general inclination of the growth trajectories (i.e., the bowtie’s degree of tilt and fanning).

**Relative aperture variance.** For an $\{a_j, b_j\}$ model, the aperture defines the point at which the conditional variance of individuals’ trajectory lines is minimized to $\hat{\sigma}_{\alpha, j}^2 \geq 0$. Drawing from the previous camera analogy, $\hat{\sigma}_{\alpha, j}^2$ is a measure reflecting the degree of dilation (vertical width) at the aperture’s opening, and it is in the squared units of the outcome measure. To derive a scale-free index of this degree of dilation, the variability must be gauged relative to some standard. Consider a standard derived from a location other than the aperture, $m_j$ metric growth units away. An $\{a_j^*, b_j\}$ model has the minimum possible $\alpha$ factor variance; models shifted by $m_j$ growth metric units to $\{a_j^* + m_j, b_j\}$, where $m_j \neq 0$, will have larger in-
intercept factor variance. Specifically, following from Equation 18 and then Equation 26
\[
\hat{\sigma}_{\alpha_j^* + m_j}^2 = \hat{\sigma}_{\alpha_j^*}^2 + 2(a_{j^*}^* - (a_{j^*}^* + m_j))\hat{\sigma}_{\alpha\beta_j^*} + [a_{j^*}^* - (a_{j^*}^* + m_j)]^2 \hat{\sigma}_{\beta_j^*}^2
\]  
(31)
\[
= \hat{\sigma}_{\alpha_j^*}^2 - 2m_j\hat{\sigma}_{\alpha\beta_j^*} + m_j^2 \hat{\sigma}_{\beta_j^*}^2
\]  
(32)
\[
= \hat{\sigma}_{\alpha_j^*}^2 + m_j^2 \hat{\sigma}_{\beta_j^*}^2
\]  
(33)

Based on this last equation, one might consider choosing a standard as the variance that occurs at a single metric growth unit away from the aperture (i.e., \(m_j = 1\)). In this case the variance of growth trajectory lines at that point becomes \(\hat{\sigma}_{\alpha_j^*}^2 + \hat{\sigma}_{\beta_j^*}^2\). Unfortunately, however, this quantity depends on the growth metric selected by the researcher. One may therefore define the standard as the variance that occurs at a point one baseline growth unit away from the aperture (to either side); that is, set \(m_j = b_j\). The variance at this point is \(\hat{\sigma}_{\alpha_j^*}^2 + b_j^2\hat{\sigma}_{\beta_j^*}^2\), and thus the relative aperture variance (RAV) may be defined as
\[
\frac{\hat{\sigma}_{\alpha_j^*}^2}{\hat{\sigma}_{\alpha_j^*}^2 + b_j^2\hat{\sigma}_{\beta_j^*}^2}
\]  
(34)

Following from Equation 17, this may also be expressed as
\[
\frac{\hat{\sigma}_{\alpha_j^*}^2}{\hat{\sigma}_{\alpha_j^*}^2 + \hat{\sigma}_{\beta_0^*}^2}
\]  
(35)

The RAV, ranging from 0 to 1, is thus a comparison of the sample’s variance at the aperture to the variance one baseline growth unit away to either side. Values nearer to 0 indicate that individuals’ growth trajectories are estimated to have a relatively strong degree of convergence at the aperture, and 0 itself would represent complete convergence. Values nearer to 1 indicate a relatively wide aperture, whereas a value of 1 could not exist because completely parallel growth trajectories (i.e., \(\hat{\sigma}_{\beta_j^*}^2 = \hat{\sigma}_{\beta_0^*}^2 = 0\)) would imply there is no aperture. Note again that this is a relative index, and thus two samples with identical aperture variance \(\hat{\sigma}_{\alpha_j^*}^2\) would have different RAV values due to differences in baseline slope variance \(\hat{\sigma}_{\beta_0^*}^2\). Visually, that is to say that for two bowties with knots of identical height, the more widely fanning tie is considered to have a smaller RAV.

**Relative gradient.** Thus far, the location and width of the aperture have been expressed in scale-free terms, and this information provides a general characteriza-
tion of the location and shape of the growth trajectories’ bowtie. As a final descriptive index, a measure of the general inclination or tilt of those trajectories is presented.

For an \( \{a_j, b_j\} \) model the slope factor has a first and second moment, estimated by \( \hat{k}_{\beta_j} \) and \( \hat{\sigma}_{\beta_j}^2 \), respectively. The estimated slope factor mean provides an average growth rate for the sample in the chosen metric, and the estimated slope factor variance gives a sense of the variability of those rates in the chosen metric (squared). If the mean is large relative to the variability, this implies that individuals have a generally similar trajectory orientation (i.e., mostly increasing or mostly decreasing); conversely, if the mean is small relative to the variability, this implies that individuals have relatively diverse trajectories (i.e., some increasing and some decreasing). Capturing these relative relations, the sample’s relative gradient (RG) may be defined as

\[
\frac{\hat{k}_{\beta_j}}{\hat{\sigma}_{\beta_j}}
\]

which is actually invariant to the choice of model \( \{a_j, b_j\} \). Given that growth rates are assumed to be distributed as \( \mathcal{N}(\hat{k}_{\beta_j}, \hat{\sigma}_{\beta_j}^2) \), the sample’s RG defines an estimated distribution \( \mathcal{N}(\text{RG}, 1) \) that may be used to facilitate description of trajectories’ slopes. Specifically, the proportion of positive slopes expected in the sample is

\[
p_{\beta}^+ = \int_0^{\infty} \Phi(\text{RG}, 1)dz
\]

where \( \Phi(\text{RG}, 1) \) is the estimated noncentral normal density function with noncentrality parameter RG. Conversely, the proportion of negative slopes is

\[
p_{\beta}^- = 1 - p_{\beta}^+ = \int_{-\infty}^0 \Phi(\text{RG}, 1)dz
\]

For example, if \( \hat{k}_{\beta_j} = 10 \) and \( \hat{\sigma}_{\beta_j}^2 = 140 \), then RG = .845. For a noncentral standard normal distribution \( \mathcal{N}(.845, 1) \), the expected proportion above 0 is \( p_{\beta}^+ = .801 \) and the expected proportion below 0 is \( p_{\beta}^- = .199 \). Thus, 80.1% of the slopes are estimated to be positive, whereas 19.9% are estimated to be negative.

Graphical Representations of RAL, RAV, and RG

The RAL, RAV, and RG provide a scale-free means to characterize the nature of a sample’s growth trajectories. In Figure 1, six illustrative diagrams are presented by
crossing two RAV conditions (large and small) with three RG conditions (positive, zero, and negative). Numerical values presented in the diagrams are for the populations from which the depicted samples were drawn. All populations have apertures occurring at the 0 point on the abscissa. Samples of size 30 were randomly drawn from each population, assuming normality of trajectories’ position at the aperture and of trajectories’ slopes.

Given the relative nature of the RAV and RG indexes, the diagrams presented in Figure 1 require no outcome variable units on the ordinate. Only a generically labeled abscissa is necessary to indicate the baseline growth units. As for the RAL, all populations had the aperture halfway through the depicted time spans (i.e., RAL = .50). However, more generally, a researcher might gather data at any shorter time span contained within the span depicted. If a researcher’s span were

<table>
<thead>
<tr>
<th></th>
<th>Large RAV</th>
<th>Small RAV</th>
</tr>
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<tbody>
<tr>
<td><strong>Positive RG</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RG : 3.333</td>
<td>RG : 1.429</td>
<td></td>
</tr>
<tr>
<td>RAV : 0.735</td>
<td>RAV : 0.338</td>
<td></td>
</tr>
<tr>
<td><strong>Zero RG</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RG : 0</td>
<td>RG : 0</td>
<td></td>
</tr>
<tr>
<td>RAV : 0.735</td>
<td>RAV : 0.338</td>
<td></td>
</tr>
<tr>
<td><strong>Negative RG</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RG : -3.333</td>
<td>RG : -1.429</td>
<td></td>
</tr>
<tr>
<td>RAV : 0.735</td>
<td>RAV : 0.338</td>
<td></td>
</tr>
</tbody>
</table>

**FIGURE 1** Illustrative linear growth trajectories as a function of RAV and RG.
entirely to the left of the aperture, the sample RAL would be > 1 as growth trajectories appear to be converging; if a researcher’s span for these samples were entirely to the right of the aperture, the sample RAL would be < 0 as growth trajectories appear to be diverging.

REAL DATA EXAMPLES

Example 1

Consider general knowledge scores from The Early Childhood Longitudinal Study, Kindergarten Class of 1998–99: ECLS–K Longitudinal Kindergarten-First Grade Public-Use Data (U.S. Department of Education, National Center for Education Statistics, 2002). The times of data collection were at approximately 6-month intervals: Fall 1998 (kindergarten), Spring 1999 (kindergarten), Fall 1999 (first grade), and Spring 2000 (first grade). The sample covariance matrix and mean vector for 4,153 children with complete data are shown here:

\[ S = \begin{bmatrix} 57.733 \\ 50.225 & 59.490 \\ 48.740 & 51.524 & 58.374 \\ 42.473 & 45.225 & 46.234 & 50.001 \end{bmatrix} \]

Means = [22.976 28.044 31.001 35.732].

Using maximum likelihood (ML) estimation within EQS 6.1 (Bentler, 2003), a standard \{0, 1\} baseline model yielded satisfactory data–model fit (e.g., comparative fit index [CFI] = .969, standardized root mean residual [SRMR] = .046). Latent parameter estimates, all of which were significant at the .05 level, were: \( \hat{\kappa}_{\alpha_0} = 23.231, \hat{\kappa}_{\beta_0} = 4.131, \hat{\sigma}_{\alpha_0}^2 = 52.881, \hat{\sigma}_{\beta_0}^2 = 0.841, \text{and} \hat{\sigma}_{\alpha\beta_0} = -2.410 \). Applying Equation 21 to these estimates, the sample’s baseline aperture shift coefficient is computed to be \( \omega_0* = -2.866 \). Using Equation 24, the estimated mean amount of \( \alpha \) at the aperture is \( \hat{\kappa}_{\alpha_0*} = 35.070 \); the estimated variance at that point (using Equation 28) is \( \hat{\sigma}_{\alpha_0*}^2 = 45.975 \). Indeed, fitting the model with loadings shifted to make the aperture as the \( \alpha \) factor reference point (i.e., with loadings of \(-2.866, -1.866, -0.866, 0.134\)) yielded \( \hat{\kappa}_{\alpha_0*} = 35.070, \hat{\sigma}_{\alpha_0*}^2 = 45.969 \) (within rounding error), and \( \hat{\sigma}_{\alpha\beta_0*} = -0.002 \) (zero within rounding error) as expected.

The RAL from Equation 20 is 0.956, indicating the sample’s aperture to be located just before the last measured time point (Spring 2000). That is, individuals are estimated to be most similar in their true amount of general knowledge just before Spring 2000. This also implies that individuals’ general knowledge growth trajectories are converging to this point for almost the entire time span examined.
Using Equation 35 the sample’s RAV is determined to be 0.982, implying relatively slow convergence to and divergence from this aperture (i.e., a relatively wide aperture). Following Equation 36, the sample’s RG is computed to be 4.505, which implies that virtually 100% of participants have general knowledge trajectories that increase over the time span examined (i.e., $\beta_{3}^{\pi} = 1.000$).

Based on these descriptives, Figure 2 depicts a set of 30 randomly generated trajectories to communicate the expected nature of individuals’ growth over time. These were drawn from a population with $\text{RAL} = 0.956$, $\text{RAV} = 0.982$, and $\text{RG} = 4.505$, assuming trajectories’ position at the aperture and slope were normally distributed. The diagram’s axes were scaled to reflect the outcome and temporal units. Also depicted is the point of aperture, and a point one baseline unit away (below), as is needed for the RAV. Overall, then, as the statistics summarized, we see largely positive general growth in general knowledge, with a fairly persistent homogeneity throughout kindergarten and first grade.

Example 2
Stoolmiller (1994) presented data on the degree of children’s Delinquent Peer Association, a score from a scale derived from the aggregated responses of a child, a parent, and a teacher. The times of data collection were fourth grade, sixth grade, seventh grade, and eighth grade. The sample covariance matrix and mean vector
for 198 children appear here (note that all scores were multiplied by 10 to avoid decimal truncation in the output):

\[
S = \begin{bmatrix}
11.000 \\
5.860 \\
6.205 \\
6.103
\end{bmatrix}
\begin{bmatrix}
13.000 \\
8.094 \\
8.798 \\
10.177
\end{bmatrix}
\begin{bmatrix}
14.000 \\
10.177 \\
16.000
\end{bmatrix}
\]

Means = [3.3 3.7 4.0 4.2].

Using ML estimation within EQS 6.1 (Bentler, 2003), a \{-4, 2\} linear latent growth model (i.e., with slope factor loadings of \(-4, -2, -1, 0\)) was fitted to the data; this sets eighth grade as the reference point, and the growth metric as annual change. This model yielded excellent data–model fit: CFI = 1.000 and SRMR = .006. Latent parameter estimates, all of which were significant at the .05 level, were: \( \hat{\kappa}_{\alpha_j} = 4.202 \), \( \hat{\kappa}_{\beta_j} = 0.229 \), \( \hat{\sigma}^2_{\alpha_j} = 11.486 \), \( \hat{\sigma}^2_{\beta_j} = 0.304 \), and \( \hat{\sigma}_{\alpha\beta_j} = 1.329 \). Applying Equation 19 to these estimates, the sample’s aperture shift coefficient is computed to be \( a^\star_j = 0.372 \). Using Equation 23, the estimated mean amount of Delinquent Peer Association at the aperture is \( \hat{\kappa}_{\alpha_j^*} = 3.201 \); the estimated variance at that point (using Equation 25) is estimated to be \( \hat{\sigma}^2_{\alpha_j^*} = 5.676 \). Indeed, fitting the model with loadings shifted to make the aperture as the \( \alpha \) factor reference point (i.e., with loadings of 0.372, 2.372, 3.372, 4.372) yielded an estimated mean of 3.202 (within rounding error), a variance of 5.677 (within rounding error), and \( \hat{\sigma}_{\alpha\beta_j^*} = -0.001 \) (zero within rounding error) as expected.

The RAL (from Equation 20) is \(-0.093\), indicating the sample’s aperture to be located just before the first measured time point (Fall 1998). This implies that from fourth through eighth grades students appear to be fanning out in terms of their associations with delinquent peers. The sample RAV is determined (using Equation 35) to be 0.949, implying as in Example 1 a relatively slow convergence to and divergence from the aperture (i.e., a relatively wide aperture). Following Equation 36, the sample’s RG is computed to be 0.415; this yields \( p^+_\beta = 0.661 \) and \( p^-_\beta = 0.339 \), which implies that approximately 66% of participants have Delinquent Peer Association trajectories that are increasing over the time span examined (and 34% are decreasing). Based on these descriptives, Figure 3 depicts a set of 30 randomly generated trajectories to communicate the expected nature of individuals’ growth over time; axes are scaled to reflect the outcome measure and temporal units. Also shown is the point of aperture, and a point one baseline unit away (above).

Thus, as we see from these trajectories, and as we would infer from the summary statistics, fourth grade might be interpreted to be somewhat of a crossroads for students. Whether arrival at this crossroads is merely the convergence of a linear process that started prior to fourth grade, or if indeed this is a critical social juncture, from this point through eighth grade students are diverging in their associations with delinquent peers. Although this divergence is relatively slow over this
span, the fact that one third of the students are expected to have a negative trajectory may indeed have implications for behavioral intervention.

Example 3

In both of the previous cases, the linearly modeled outcome was a measured variable. However, growth in a latent variable is often of interest as well. In what has been called a curve of factors model (McArdle, 1988) and a second-order latent growth model (Hancock, Kuo, & Lawrence, 2001; Sayer & Cumsille, 2001), a common set of indicator variables is measured at \( t \) time points. These variables indicate a factor \( \eta \) at each of the time points, with a measurement model generally assumed to be invariant across time. These first-order factors in vector \( \eta \) are, in turn, modeled as dependent on the second-order growth factors in vector \( \xi \). For linear models \( \eta = \Gamma \xi + \zeta \), where \( \Gamma \) is the analog to \( \Lambda \) from the first-order linear growth model.

Hancock et al. (2001) presented data on 791 adolescent females’ self-concept at 8th, 10th, and 12th grade, as indicated by three questionnaire items at each time point. These data were drawn from the National Education Longitudinal Survey of 1988 data set (NELS:88), sponsored by the National Center for Education Statistics, U.S. Department of Education (see Ingels et al., 1994). The sample covariance matrix and mean vector appear here, where \( X_1, X_4, \) and \( X_7 \) are the same measure at

![FIGURE 3  Expected linear growth trajectories for Delinquent Peer Association.](image-url)
different time points; $X_2$, $X_5$, and $X_8$ are the same measure at different time points; and $X_3$, $X_6$, and $X_9$ are the same measure at different time points:

$$\begin{bmatrix}
.865 \\
.471 & 1.210 \\
.564 & .584 & 1.346 \\
.249 & .215 & .267 & 1.769 \\
.194 & .223 & .250 & 1.545 & 2.016 \\
.174 & .203 & .265 & 1.555 & 1.652 & 1.960 \\
.141 & .097 & .260 & .253 & .279 & .307 & 2.560 \\
.163 & .167 & .273 & .227 & .311 & .306 & 2.314 & 2.789 \\
.117 & .093 & .264 & .215 & .293 & .326 & 2.344 & 2.312 & 2.723
\end{bmatrix}$$

Means = [3.10 3.17 3.00 2.96 2.97 2.81 2.88 2.91 2.80].

Using ML estimation within EQS 6.1 (Bentler, 2003), a \{0, 2\} linear latent growth model (i.e., with slope factor loadings of 0, 2, 4) was fitted to the data; this sets eighth grade as the reference point, and the growth metric as annual change (note that Hancock et al., 2001, used a \{0, 1\} model). Corresponding first-order factor loadings and intercepts were constrained to be equal as originally done, using the first indicator of each factor ($X_1$, $X_4$, and $X_7$) as the scale indicator. This model yielded excellent data–model fit: CFI = .998 and SRMR = .027. Latent parameter estimates (drawing from the indicator variables’ metric) were $\hat{\beta}_{\alpha} = 3.102$, $\hat{\beta}_{\beta} = -0.061$, both of which were statistically significant at the .05 level, and $\hat{\sigma}^2_{\alpha} = 0.256$, $\hat{\sigma}^2_{\beta} = 0.019$, and $\hat{\sigma}_{\alpha\beta} = 0.024$, only the first of which was statistically significant at the .05 level. Applying Equation 19 to these estimates, the sample’s aperture shift coefficient is computed to be $a_j^* = -1.263$. Using Equation 23, the estimated mean amount of $\eta$ at the aperture (in the indicator variables’ metric) is $\hat{\eta} = 3.025$; the estimated variance at that point (using Equation 25) is estimated to be $\hat{\sigma}^2_{\eta,j} = 0.226$. Indeed, fitting the model with loadings in $\Gamma$ shifted to make the aperture as the $\alpha$ factor reference point (i.e., with loadings of $-1.263$, 0.737, 2.737) yielded an estimated mean of 3.026 (within rounding error), a variance of 0.225 (within rounding error), and $\hat{\sigma}^2_{\eta,j} = 0.000$ as expected.

The RAL (from Equation 20) is 0.316, which implies that individuals are estimated to be most similar in their amount of latent self-concept between the 9th and 10th grades. The sample RAV is determined (using Equation 34) to be 0.771, implying a tighter aperture than the two previous examples. Following Equation 36, the sample’s RG is computed to be $-0.443$; this yields $p_{\beta}^+ = 0.329$ and $p_{\beta}^- = 0.671$, which implies that approximately 67% of participants have latent self-concept trajectories that are decreasing over the time span examined (and 33%
are increasing). Based on these descriptives, Figure 4 depicts a set of 30 randomly generated trajectories to communicate the expected nature of individuals’ growth over in latent self-concept time. The abscissa is labeled with the temporal metric, whereas the ordinate is left unlabeled as the latent outcome has no inherent metric. Also depicted is the point of aperture, and a point one baseline unit away (below).

Inspection of these sample latent trajectories, and following from the statistics presented, the following characterization of females’ latent self-concept results. Given the reasonableness of a linear model, females appear to be quite diverse in their self-concept amounts and trajectories from 8th through 12th grade. Although they reach a point of relative homogeneity soon after 9th grade, typically the start of high school, approximately twice as many are decreasing as are increasing at that point of aperture. Identifying the determinants of this vast number in decline would seem to be of critical importance to educational and counseling psychologists.

**SUMMARY AND CONCLUSIONS**

This article makes two primary contributions to latent growth modeling. First, a nomenclature is synthesized from recent literature to describe imposed linear models using the specified $\alpha$ factor reference point and $\beta$ factor metric. As mentioned previously, both applied and technical researchers should find this simple system useful to communicate the nature of models of theoretical or methodologi-
Having illustrated how latent parameter estimates alter as a function of those specifications (a relation which, for the covariance structure, is tantamount to factor rotation within a growth modeling framework), the second primary contribution becomes possible. Specifically, several descriptive statistics are recommended as a way to characterize the nature of the linear latent growth trajectories arising from sample data, whether those trajectories involve measured or latent outcomes. For applied researchers, this means communicating a clearer characterization of where along the time continuum their participants are developmentally most similar (RAL), how homogeneous those participants are at that point (RAV), and the direction of the participants’ change over the time interval studied (RG, $p^\beta_1$ and $p^\beta_2$). Meanwhile, for methodological researchers examining model behavior under a variety of conditions that might occur in applied settings, a range of values for RAL, RAV, and RG could be crossed factorially to define the scope of data simulation conditions. For example, simulated data could be manipulated to yield some or all of the following RAL values: $-0.50$ (aperture below measured time span), $0$ (aperture at first time point), $0.50$ (aperture at middle time point), $1.00$ (aperture at last time point), and $1.50$ (aperture above measured time span). The RAV values could simultaneously be manipulated to be $0.25$ (relatively small), $0.50$ (moderate), or $0.75$ (relatively large), and RG levels could be set at $-2.00$ (negative, steep slope), $-1.00$ (negative, gentle slope), $0$ (zero average slope), $1.00$ (positive, gentle slope), and $2.00$ (positive, steep slope).

Finally, further investigation regarding the methods presented here could proceed along a number of fronts. Pertaining to the notational scheme for describing model structure, extensions of these ideas to more complex growth models (e.g., nonlinear, multiple domains) could prove most useful. Additionally, the descriptive statistics proposed in this article could be accompanied by confidence intervals (derivable using, e.g., LISREL’s additional parameters feature), and hypothesis tests could be developed within multisample growth modeling frameworks to test for population differences in these characteristics. This latter application might be particularly interesting in multisample growth models involving interventions (see, e.g., Muthén & Curran, 1997), thereby assessing the impact of intervention on population RAL, RAV, and RG characteristics. Potential also exists for detecting mixtures of populations having different RAL, RAV, and RG, extending the latent growth mixture modeling literature (see, e.g., Muthén, 2004). Such developments are eagerly awaited.

REFERENCES


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